An Exact Identifiability of Variables in General Linear Problem of Input–Output Analysis

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A general problem of input–output analysis is considered in this study as a system of equations written in terms of free variables for any rectangular supply and use table given. This system spans the regular linear equations for material and financial balances, a batch of predetermined values for exogenous variables chosen in advance and an additional set of linkage equations that provides the exact identifiability for all unknown variables. General problem is linear if all of its equations is linear, and becomes nonlinear otherwise.

The paper is concerned with some operational opportunities for constructing a set of the identifying linear equations in the cases of evaluating the response of the economy to exogenous changes in final demand and value added vectors. Toward this end, it is expedient to involve into consideration the matrix-valued linear cost and production functions of product and industry outputs and inputs as their arguments respectively. Moreover, the product-mix and market shares contours of supply and use matrices appear to be operational and motivated in the circumstances.

Feasible transformations for general linear problem of input–output analysis within the involved linear equations are studied. As a result, eight different specifications of general linear input–output problem arise under various conditions for exact identifiability of unknown variables. Two of them form an algebraic framework of Leontief demand-driven quantity and relative supply-driven price models, whereas the other two provide an algebraic foundation for compiling Ghosh supply-driven price and relative demand-driven quantity models.

For each specification, the regular solution and the supplementary solution (corresponding to a situation when numbers of products and sectors coincide) are obtained in matrix notation. It is shown that some solutions seem to be implausible artefacts that are out of economic sense (these implausible solutions generate an instrumental framework for an industry technology assumption and a fixed product sales structure assumption, which are widely used in the transformation of supply and use tables to symmetric input-output tables). The formal linkage between Leontief and Ghosh quantity and price models for symmetric input-output tables is established.

Keywords: rectangular supply and use table, matrix-valued production and cost functions, exogenous changes in final demand and value added, demand-driven and supply-driven models, quantity and price models

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1. A general linear problem of input–output analysis

Any supply and use table for certain time period (say, period 0) is generated by a pair of rectangular matrices: supply, or production, matrix $\mathbf{X}_0$ and use for intermediates, or intermediate consumption, matrix $\mathbf{Z}_0$ of dimension $N\times M$ both where N is a number of commodities, or products, and M is a number of sectors, or industries, in the economy under consideration. In mathematical notation, supply and use table is determined by the vector equation for material balance of products' intermediate and final uses

$$\mathbf{X}_0\mathbf{e}_M = \mathbf{Z}_0\mathbf{e}_M + \mathbf{y}_0 \tag{1}$$

and by the vector equation for financial balance of industries' intermediate and primary (combined into value added) inputs

$$\mathbf{e}'_N\mathbf{X}_0 = \mathbf{e}'_N\mathbf{Z}_0 + \mathbf{v}'_0 \tag{2}$$

where $\mathbf{e}_N$ and $\mathbf{e}_M$ are $N\times1$ and $M\times1$ summation column vectors with unit elements, $\mathbf{y}_0$ is a column vector of net final demand with dimensions $N\times1$, and $\mathbf{v}_0$ is a column vector of value added with dimensions $M\times1$. Here putting a prime after vector's or matrix's symbol denotes a transpose of this vector or matrix.

For analytical purposes, one needs to rewrite the system of balance equations (1), (2) in terms of free variables. Let $\mathbf{x}_\downarrow = \mathbf{X}\mathbf{e}_M$ be N-dimensional column vector of product outputs, and $\mathbf{x}'_\rightarrow = \mathbf{e}'_N\mathbf{X}$ be $1\times M$ row vector of industry outputs. Also, let $\mathbf{z}_\downarrow = \mathbf{Z}\mathbf{e}_M$ be $N\times1$ column vector of product amounts in intermediate consumptions, and $\mathbf{z}'_\rightarrow = \mathbf{e}'_N\mathbf{Z}$ be M-dimensional row vector of industry expenditures for intermediate consumption. The vectors $\mathbf{x}_\downarrow, \mathbf{x}_\rightarrow, \mathbf{z}_\downarrow, \mathbf{z}_\rightarrow$ are sometimes called product and industry (column and row) marginal totals for the production matrix $\mathbf{X}$ and the intermediate consumption matrix $\mathbf{Z}$. Thus, the system of N+M scalar equations (1), (2) can be written in free vector variables as follows:

$$\mathbf{x}_\downarrow = \mathbf{z}_\downarrow + \mathbf{y}, \qquad \mathbf{x}'_\rightarrow = \mathbf{z}'_\rightarrow + \mathbf{v}'. \tag{3}$$

The aim of constructing similar balance models is to assess an impact of exogenous (absolute or relative) changes in final demand (or, generally, in other product variables) and, by certain symmetry of balance equations considered, exogenous changes in value added (or, generally, in other industry variables) on the economy. Balance models do not usually reflect the true causes of the certain changes in final demand or value added, so a response of the economy to any exogenous perturbation is evaluated in the mode of getting answers to questions like "what would happen if ...? ".

The balance model (3) contains N+M linear equations with 3(N+M) scalar variables. Assume that exogenous perturbation is expressed in terms of k exogenous variables. To provide exact (or strict) identifiability of the model it is required to incorporate into the model $2(N+M) - k$ auxiliary independent equations as a certain set of linkages between the variables. In particular, $N+2M$ independent equations are needed at $k = N$, and $2N+M$ equations are needed at $k = M$. The structure of initial supply and use table serves as an informational framework for constructing the auxiliary linkage equations.

In this context, a general linear problem of input–output analysis is considered as the system of equations (3) together with a chosen specification of exogenous perturbation and a corresponding set of linear linkages between the variables, which provides the strict identification of all unknown variables. Note that the general problem of input–output analysis becomes nonlinear if at least one of its equations is nonlinear.

2. The specifications of linear linkages between the variables

Let us study some operational opportunities for constructing a set of identifying linear equations in the cases of evaluating the response of the economy to exogenous changes in the final demand vector $\mathbf{y} = \mathbf{y}^* \neq \mathbf{y}_0$ with dimensions $N\times1$ or the value added vector $\mathbf{v} = \mathbf{v}^* \neq \mathbf{v}_0$ with dimensions $M\times1$.

To this end, first, one can introduce a pair of matrix-valued cost functions

$$\mathbf{Z} = \mathbf{A}_{\downarrow}\hat{\mathbf{x}}_{\rightarrow}, \qquad \mathbf{A}_{\downarrow} = \mathbf{Z}_0\langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}; \tag{4}$$

$$\mathbf{Z} = \hat{\mathbf{x}}_{\downarrow}\mathbf{A}_{\rightarrow}, \qquad \mathbf{A}_{\rightarrow} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1}\mathbf{Z}_0 \tag{5}$$

where $\mathbf{A}_{\downarrow}$ and $\mathbf{A}_{\rightarrow}$ are the known $N\times M$ matrices of relative coefficients, and angled bracketing around a vector's symbol (or putting a "hat" over it) denotes a diagonal matrix, with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697). Statements (4) and (5) postulate two specifications for linear dependency of intermediate consumption matrix $\mathbf{Z}$ from industry output vector $\mathbf{x}_{\rightarrow}$ and product output vector $\mathbf{x}_{\downarrow}$, respectively.

Secondly, a mirror-image pair of matrix-valued production functions

$$\mathbf{X} = \mathbf{B}_{\downarrow}\hat{\mathbf{z}}_{\rightarrow}, \qquad \mathbf{B}_{\downarrow} = \mathbf{X}_0\langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1}; \tag{6}$$

$$\mathbf{X} = \hat{\mathbf{z}}_{\downarrow}\mathbf{B}_{\rightarrow}, \qquad \mathbf{B}_{\rightarrow} = \langle \mathbf{Z}_0 \mathbf{e}_M \rangle^{-1}\mathbf{X}_0 \tag{7}$$

seems to be helpful for our analytical purposes. Here $\mathbf{B}_{\downarrow}$ and $\mathbf{B}_{\rightarrow}$ are also the known $N\times M$ matrices of relative coefficients. Statements (6) and (7) involve two specifications for linear dependency of production matrix $\mathbf{X}$ from vector of industry expenditures for intermediate consumption $\mathbf{z}_{\rightarrow}$ and vector of product amounts in intermediate consumption $\mathbf{z}_{\downarrow}$, respectively.

Finally, a quite explainable requirement to keep invariable a "vertical" (or a "horizontal") structure of production (or intermediate consumption) matrix in terms of its column (or row) marginal totals leads to

$$\mathbf{X} = \mathbf{C}_{\downarrow}\hat{\mathbf{x}}_{\rightarrow}, \qquad \mathbf{C}_{\downarrow} = \mathbf{X}_0\langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}; \tag{8}$$

$$\mathbf{X} = \hat{\mathbf{x}}_{\downarrow}\mathbf{C}_{\rightarrow}, \qquad \mathbf{C}_{\rightarrow} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1}\mathbf{X}_0; \tag{9}$$

$$\mathbf{Z} = \mathbf{D}_{\downarrow}\hat{\mathbf{z}}_{\rightarrow}, \qquad \mathbf{D}_{\downarrow} = \mathbf{Z}_0\left\langle \mathbf{e}'_N \mathbf{Z}_0 \right\rangle^{-1}; \tag{10}$$

$$\mathbf{Z} = \hat{\mathbf{z}}_{\downarrow}\mathbf{D}_{\rightarrow}, \qquad \mathbf{D}_{\rightarrow} = \left\langle \mathbf{Z}_0 \mathbf{e}_M \right\rangle^{-1}\mathbf{Z}_0 \tag{11}$$

where $\mathbf{C}_{\downarrow}$, $\mathbf{C}_{\rightarrow}$, $\mathbf{D}_{\downarrow}$ and $\mathbf{D}_{\rightarrow}$ are the known N×M matrices of relative coefficients, as earlier. These "structural" equations closes a full set of linear linkages between the variables in model (3) under consideration.

3. Feasible transformations for general linear problem of input–output analysis

It is easy to see that the linear equations (4) – (11) cannot be used for further transformations of balance model (3) simultaneously. Therefore, it is necessary to define the order, in which they may be applied.

Matrix-valued cost functions (4) and (5) allow to eliminate the intermediate input marginal totals $\mathbf{z}_{\rightarrow}$ and $\mathbf{z}_{\downarrow}$ from balance model (3) in two ways, whereas matrix-valued production functions (6) and (7) provide a two-way removal of the output marginal totals $\mathbf{x}_{\rightarrow}$ and $\mathbf{x}_{\downarrow}$ from (3). Hence, we get two models, namely (3), (4) and (3), (5), each of which comprises N+M linear equations with 2(N+M) scalar variables $\mathbf{x}_{\downarrow}, \mathbf{x}_{\rightarrow}, \mathbf{y}, \mathbf{v}$, and a pair of models, namely (3), (6) and (3), (7), each of which also contains N+M linear equations with 2(N+M) scalar variables $\mathbf{z}_{\downarrow}, \mathbf{z}_{\rightarrow}, \mathbf{y}, \mathbf{v}$.

Further, one can apply

- equation (8) as $\mathbf{x}_{\downarrow} = \mathbf{C}_{\downarrow}\mathbf{x}_{\rightarrow}$ to delete vector $\mathbf{x}_{\downarrow}$ from models (3), (4) and (3), (5),
- equation (9) as $\mathbf{x}_{\rightarrow} = \mathbf{C}'_{\rightarrow}\mathbf{x}_{\downarrow}$ to eliminate $\mathbf{x}_{\rightarrow}$ from the same models,
- equation (10) as $\mathbf{z}_{\downarrow} = \mathbf{D}_{\downarrow}\mathbf{z}_{\rightarrow}$ for removal of vector $\mathbf{z}_{\downarrow}$ from models (3), (6) and (3), (7),
- and, finally, equation (11) as $\mathbf{z}_{\rightarrow} = \mathbf{D}'_{\rightarrow}\mathbf{z}_{\downarrow}$ for deleting $\mathbf{z}_{\rightarrow}$ from the latter models.

As a result, we successively obtain eight following different specifications of general linear input–output analysis problem:

[(3), (4), (8)] $$\mathbf{C}_{\downarrow}\mathbf{x}_{\rightarrow} = \mathbf{A}_{\downarrow}\mathbf{x}_{\rightarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\rightarrow} = \mathbf{x}'_{\rightarrow}\left\langle \mathbf{e}'_N \mathbf{A}_{\downarrow} \right\rangle + \mathbf{v}'; \tag{12}$$

[(3), (4), (9)] $$\mathbf{x}_{\downarrow} = \mathbf{A}_{\downarrow}\mathbf{C}'_{\rightarrow}\mathbf{x}_{\downarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\downarrow}\mathbf{C}_{\rightarrow} = \mathbf{x}'_{\downarrow}\mathbf{C}_{\rightarrow}\left\langle \mathbf{e}'_N \mathbf{A}_{\downarrow} \right\rangle + \mathbf{v}'; \tag{13}$$

[(3), (5), (8)] $$\mathbf{C}_{\downarrow}\mathbf{x}_{\rightarrow} = \left\langle \mathbf{A}_{\rightarrow}\mathbf{e}_M \right\rangle \mathbf{C}_{\downarrow}\mathbf{x}_{\rightarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\rightarrow} = \mathbf{x}'_{\rightarrow}\mathbf{C}'_{\downarrow}\mathbf{A}_{\rightarrow} + \mathbf{v}'; \tag{14}$$

[(3), (5), (9)] $$\mathbf{x}_{\downarrow} = \left\langle \mathbf{A}_{\rightarrow}\mathbf{e}_M \right\rangle \mathbf{x}_{\downarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\downarrow}\mathbf{C}_{\rightarrow} = \mathbf{x}'_{\downarrow}\mathbf{A}_{\rightarrow} + \mathbf{v}'; \tag{15}$$

[(3), (6), (10)] $$\mathbf{B}_{\downarrow}\mathbf{z}_{\rightarrow} = \mathbf{D}_{\downarrow}\mathbf{z}_{\rightarrow} + \mathbf{y}, \qquad \mathbf{z}'_{\rightarrow}\left\langle \mathbf{e}'_N \mathbf{B}_{\downarrow} \right\rangle = \mathbf{z}'_{\rightarrow} + \mathbf{v}'; \tag{16}$$

$$[(3), (6), (11)] \qquad \mathbf{B}_{\downarrow}\mathbf{D}'_{\rightarrow}\mathbf{z}_{\downarrow} = \mathbf{z}_{\downarrow} + \mathbf{y}\,, \qquad \mathbf{z}'_{\downarrow}\mathbf{D}_{\rightarrow}\left\langle \mathbf{e}'_{N}\mathbf{B}_{\downarrow} \right\rangle = \mathbf{z}'_{\downarrow}\mathbf{D}_{\rightarrow} + \mathbf{v}'\,; \tag{17}$$

$$[(3), (7), (10)] \qquad \left\langle \mathbf{B}_{\rightarrow}\mathbf{e}_{M} \right\rangle \mathbf{D}_{\downarrow}\mathbf{z}_{\rightarrow} = \mathbf{D}_{\downarrow}\mathbf{z}_{\rightarrow} + \mathbf{y}\,, \qquad \mathbf{z}'_{\rightarrow}\mathbf{D}'_{\downarrow}\mathbf{B}_{\rightarrow} = \mathbf{z}'_{\rightarrow} + \mathbf{v}'\,; \tag{18}$$

$$[(3), (7), (11)] \qquad \left\langle \mathbf{B}_{\rightarrow}\mathbf{e}_{M} \right\rangle \mathbf{z}_{\downarrow} = \mathbf{z}_{\downarrow} + \mathbf{y}\,, \qquad \mathbf{z}'_{\downarrow}\mathbf{B}_{\rightarrow} = \mathbf{z}'_{\downarrow}\mathbf{D}_{\rightarrow} + \mathbf{v}'\,. \tag{19}$$

Note that each model specification consists of N+M linear equations with different numbers of unknown scalar variables, namely N+2M or 2N+M.

If the number of unknown variables in a certain model equals N+2M, supplementing an exogenous condition $\mathbf{v} = \mathbf{v}_*$ provides a just identifying closure of this model. However, if the values N and M coincide, alternative exogenous choice of condition $\mathbf{y} = \mathbf{y}_*$ appears to be also feasible. All possible cases for models (12) – (19) are represented in Table 1.

Table 1. The various specifications of linear input–output analysis problem with exogenous variables

Model	Model code	Matrices fixed	Vector variables	Number of variables	Exogenous variable	Alternative exogenous variable at $N = M$
(12)	$\mathbf{A}_{\downarrow}\mathbf{C}_{\downarrow}$	$\mathbf{A}_{\downarrow}, \mathbf{C}_{\downarrow}$	$\mathbf{x}_{\rightarrow}, \mathbf{y}, \mathbf{v}$	N+2M	$\mathbf{v} = \mathbf{v}_*$	$\mathbf{y} = \mathbf{y}_*$
(13)	$\mathbf{A}_{\downarrow}\mathbf{C}_{\rightarrow}$	$\mathbf{A}_{\downarrow}, \mathbf{C}_{\rightarrow}$	$\mathbf{x}_{\downarrow}, \mathbf{y}, \mathbf{v}$	2N+M	$\mathbf{y} = \mathbf{y}_*$	$\mathbf{v} = \mathbf{v}_*$
(14)	$\mathbf{A}_{\rightarrow}\mathbf{C}_{\downarrow}$	$\mathbf{A}_{\rightarrow}, \mathbf{C}_{\downarrow}$	$\mathbf{x}_{\rightarrow}, \mathbf{y}, \mathbf{v}$	N+2M	$\mathbf{v} = \mathbf{v}_*$	$\mathbf{y} = \mathbf{y}_*$
(15)	$\mathbf{A}_{\rightarrow}\mathbf{C}_{\rightarrow}$	$\mathbf{A}_{\rightarrow}, \mathbf{C}_{\rightarrow}$	$\mathbf{x}_{\downarrow}, \mathbf{y}, \mathbf{v}$	2N+M	$\mathbf{y} = \mathbf{y}_*$	$\mathbf{v} = \mathbf{v}_*$
(16)	$\mathbf{B}_{\downarrow}\mathbf{D}_{\downarrow}$	$\mathbf{B}_{\downarrow}, \mathbf{D}_{\downarrow}$	$\mathbf{z}_{\rightarrow}, \mathbf{y}, \mathbf{v}$	N+2M	$\mathbf{v} = \mathbf{v}_*$	$\mathbf{y} = \mathbf{y}_*$
(17)	$\mathbf{B}_{\downarrow}\mathbf{D}_{\rightarrow}$	$\mathbf{B}_{\downarrow}, \mathbf{D}_{\rightarrow}$	$\mathbf{z}_{\downarrow}, \mathbf{y}, \mathbf{v}$	2N+M	$\mathbf{y} = \mathbf{y}_*$	$\mathbf{v} = \mathbf{v}_*$
(18)	$\mathbf{B}_{\rightarrow}\mathbf{D}_{\downarrow}$	$\mathbf{B}_{\rightarrow}, \mathbf{D}_{\downarrow}$	$\mathbf{z}_{\rightarrow}, \mathbf{y}, \mathbf{v}$	N+2M	$\mathbf{v} = \mathbf{v}_*$	$\mathbf{y} = \mathbf{y}_*$
(19)	$\mathbf{B}_{\rightarrow}\mathbf{D}_{\rightarrow}$	$\mathbf{B}_{\rightarrow}, \mathbf{D}_{\rightarrow}$	$\mathbf{z}_{\downarrow}, \mathbf{y}, \mathbf{v}$	2N+M	$\mathbf{y} = \mathbf{y}_*$	$\mathbf{v} = \mathbf{v}_*$

Thus, in terms of exogenous final demand and value added, each input–output model (12) – (19) has two solutions – a regular one and a supplementary one with an alternative exogenous vector at $N = M$.

4. Regular solutions for the specifications of linear input–output analysis problem

It is not so difficult to show that any regular solution for models (12) – (19) can be written in a following common form:

$$\mathbf{X} = \hat{\mathbf{p}}_{\mathrm{X}}\mathbf{X}_0\hat{\mathbf{q}}_{\mathrm{X}}\,, \qquad \mathbf{Z} = \hat{\mathbf{p}}_{\mathrm{Z}}\mathbf{Z}_0\hat{\mathbf{q}}_{\mathrm{Z}} \tag{20}$$

where $\mathbf{p}_{\mathrm{X}}$, $\mathbf{p}_{\mathrm{Z}}$ are the computable vectors with dimensions N×1, and $\mathbf{q}_{\mathrm{X}}$, $\mathbf{q}_{\mathrm{Z}}$ are the computable vectors with dimensions $M\times 1$. It seems to be a unique way to interpret this result, namely **p**'s and **q**'s should be considered as the relative price indices and the relative volume (quantity) indices respectively.

All regular solutions for models (12) – (19) in terms of the column vectors from (20) are grouped in Table 2 with denotation for relative coefficients of final demand and value added as follows:

$$\boldsymbol{y}^*_{\mathbf{y}} = \hat{\mathbf{y}}_0^{-1}\mathbf{y}_*\,, \qquad \boldsymbol{y}^*_{\mathbf{x}} = \left\langle \mathbf{X}_0\mathbf{e}_M \right\rangle^{-1}\mathbf{y}_*\,, \qquad \boldsymbol{y}^*_{\mathbf{z}} = \left\langle \mathbf{Z}_0\mathbf{e}_M \right\rangle^{-1}\mathbf{y}_*\,;$$

$$\boldsymbol{v}^*_{\mathbf{v}} = \hat{\mathbf{v}}_0^{-1}\mathbf{v}_*\,, \qquad \boldsymbol{v}^*_{\mathbf{x}} = \left\langle \mathbf{e}'_N\mathbf{X}_0 \right\rangle^{-1}\mathbf{v}_*\,, \qquad \boldsymbol{v}^*_{\mathbf{z}} = \left\langle \mathbf{e}'_N\mathbf{Z}_0 \right\rangle^{-1}\mathbf{v}_*\,.$$

Table 2. Regular solutions for the specifications of linear input–output analysis problem

Model	Model code	Formulae for vectors $\mathbf{p}_\mathbf{X}$, $\mathbf{q}_\mathbf{X}$	Formulae for vectors $\mathbf{p}_\mathbf{Z}$, $\mathbf{q}_\mathbf{Z}$	Exogenous variable
(12)	$\mathbf{A}_{\downarrow}\mathbf{C}_{\downarrow}$	$\mathbf{p}_\mathbf{X} = \mathbf{e}_N$, $\mathbf{q}_\mathbf{X} = \boldsymbol{v}^*_{\mathbf{v}}$	$\mathbf{p}_\mathbf{Z} = \mathbf{e}_N$, $\mathbf{q}_\mathbf{Z} = \mathbf{q}_\mathbf{X}$	$\mathbf{v} = \mathbf{v}_*$
(13)	$\mathbf{A}_{\downarrow}\mathbf{C}_{\rightarrow}$	$\mathbf{p}_\mathbf{X} = \left(\mathbf{E}_N - \mathbf{A}_{\rightarrow}\mathbf{C}'_{\downarrow}\right)^{-1}\boldsymbol{y}^*_{\mathbf{x}}$, $\mathbf{q}_\mathbf{X} = \mathbf{e}_M$	$\mathbf{p}_\mathbf{Z} = \mathbf{E}_N$, $\mathbf{q}_\mathbf{Z} = \mathbf{C}'_{\downarrow}\mathbf{p}_\mathbf{X}$	$\mathbf{y} = \mathbf{y}_*$
(14)	$\mathbf{A}_{\rightarrow}\mathbf{C}_{\downarrow}$	$\mathbf{p}_\mathbf{X} = \mathbf{e}_N$, $\mathbf{q}_\mathbf{X} = \left(\mathbf{E}_M - \mathbf{A}'_{\downarrow}\mathbf{C}_{\rightarrow}\right)^{-1}\boldsymbol{v}^*_{\mathbf{x}}$	$\mathbf{p}_\mathbf{Z} = \mathbf{C}_{\rightarrow}\mathbf{q}_\mathbf{X}$, $\mathbf{q}_\mathbf{Z} = \mathbf{e}_M$	$\mathbf{v} = \mathbf{v}_*$
(15)	$\mathbf{A}_{\rightarrow}\mathbf{C}_{\rightarrow}$	$\mathbf{p}_\mathbf{X} = \boldsymbol{y}^*_{\mathbf{y}}$, $\mathbf{q}_\mathbf{X} = \mathbf{e}_M$	$\mathbf{p}_\mathbf{Z} = \mathbf{p}_\mathbf{X}$, $\mathbf{q}_\mathbf{Z} = \mathbf{e}_M$	$\mathbf{y} = \mathbf{y}_*$
(16)	$\mathbf{B}_{\downarrow}\mathbf{D}_{\downarrow}$	$\mathbf{p}_\mathbf{X} = \mathbf{e}_N$, $\mathbf{q}_\mathbf{X} = \boldsymbol{v}^*_{\mathbf{v}}$	$\mathbf{p}_\mathbf{Z} = \mathbf{e}_N$, $\mathbf{q}_\mathbf{Z} = \mathbf{q}_\mathbf{X}$	$\mathbf{v} = \mathbf{v}_*$
(17)	$\mathbf{B}_{\downarrow}\mathbf{D}_{\rightarrow}$	$\mathbf{p}_\mathbf{X} = \mathbf{e}_N$, $\mathbf{q}_\mathbf{X} = \mathbf{D}'_{\downarrow}\mathbf{p}_\mathbf{Z}$	$\mathbf{p}_\mathbf{Z} = \left(\mathbf{B}_{\rightarrow}\mathbf{D}'_{\downarrow} - \mathbf{E}_N\right)^{-1}\boldsymbol{y}^*_{\mathbf{z}}$, $\mathbf{q}_\mathbf{Z} = \mathbf{e}_M$	$\mathbf{y} = \mathbf{y}_*$
(18)	$\mathbf{B}_{\rightarrow}\mathbf{D}_{\downarrow}$	$\mathbf{p}_\mathbf{X} = \mathbf{D}_{\rightarrow}\mathbf{q}_\mathbf{Z}$, $\mathbf{q}_\mathbf{X} = \mathbf{e}_M$	$\mathbf{p}_\mathbf{Z} = \mathbf{e}_N$, $\mathbf{q}_\mathbf{Z} = \left(\mathbf{B}'_{\downarrow}\mathbf{D}_{\rightarrow} - \mathbf{E}_M\right)^{-1}\boldsymbol{v}^*_{\mathbf{z}}$	$\mathbf{v} = \mathbf{v}_*$
(19)	$\mathbf{B}_{\rightarrow}\mathbf{D}_{\rightarrow}$	$\mathbf{p}_\mathbf{X} = \boldsymbol{y}^*_{\mathbf{y}}$, $\mathbf{q}_\mathbf{X} = \mathbf{e}_M$	$\mathbf{p}_\mathbf{Z} = \mathbf{p}_\mathbf{X}$, $\mathbf{q}_\mathbf{Z} = \mathbf{e}_M$	$\mathbf{y} = \mathbf{y}_*$

Note that $\mathbf{E}_N = \hat{\mathbf{e}}_N$ and $\mathbf{E}_M = \hat{\mathbf{e}}_M$ are the identity matrices of order N and M respectively. It is interesting to see the regular solutions for models $\mathbf{A}_{\downarrow}\mathbf{C}_{\downarrow}$ (12) and $\mathbf{B}_{\downarrow}\mathbf{D}_{\downarrow}$ (16) pairwise coincided as well as the regular solutions for models $\mathbf{A}_{\rightarrow}\mathbf{C}_{\rightarrow}$ (15) and $\mathbf{B}_{\rightarrow}\mathbf{D}_{\rightarrow}$ (19). Besides, only one vector from each pair $\mathbf{p}_\mathbf{X}$, $\mathbf{q}_\mathbf{X}$ does not equal $\mathbf{e}_N$ or $\mathbf{e}_M$. In the same way there is a unique vector in each pair $\mathbf{p}_\mathbf{Z}$, $\mathbf{q}_\mathbf{Z}$ that is distinct from $\mathbf{e}_N$ or $\mathbf{e}_M$.

5. Supplementary solutions for the specifications of linear input–output analysis problem

In order to distinguish matrix notation in the regular case and in the case of $N = M = K$ we will use the same matrix symbols with underscore in latter case. Any supplementary solution for models (12) – (19) with the alternative exogenous vectors can be also representing as

$$\underline{\mathbf{X}} = \hat{\mathbf{p}}_\mathbf{X}\underline{\mathbf{X}}_0\hat{\mathbf{q}}_\mathbf{X}\,, \qquad \underline{\mathbf{Z}} = \hat{\mathbf{p}}_\mathbf{Z}\underline{\mathbf{Z}}_0\hat{\mathbf{q}}_\mathbf{Z} \tag{21}$$

where $\mathbf{p}_{\mathrm{X}}$,$\mathbf{p}_{\mathrm{Z}}$ and $\mathbf{q}_{\mathrm{X}}$,$\mathbf{q}_{\mathrm{Z}}$ are the column vectors with dimensions $K\times 1$. As earlier, all **p**'s and **q**'s are interpreted as the relative price indices and the relative volume (quantity) indices respectively.

Supplementary solutions for models (12) – (19) in terms of the column vectors from (21) are gathered in Table 3.

Table 3. Supplementary solutions for the specifications of linear input–output analysis problem

Model	Model code	Formulae for vectors $\mathbf{p}_{\mathrm{X}}$, $\mathbf{q}_{\mathrm{X}}$	Formulae for vectors $\mathbf{p}_{\mathrm{Z}}$,$\mathbf{q}_{\mathrm{Z}}$	Exogenous variable
(12)	$\underline{\mathbf{A}}_{\downarrow}\underline{\mathbf{C}}_{\downarrow}$	$\mathbf{p}_{\mathrm{X}}=\mathbf{e}_K$, $\mathbf{q}_{\mathrm{X}}=(\underline{\mathbf{X}}_0-\underline{\mathbf{Z}}_0)^{-1}\mathbf{y}_*$	$\mathbf{p}_{\mathrm{Z}}=\mathbf{e}_K$, $\mathbf{q}_{\mathrm{Z}}=\mathbf{q}_{\mathrm{X}}$	$\mathbf{y}=\mathbf{y}_*$
(13)	$\underline{\mathbf{A}}_{\downarrow}\underline{\mathbf{C}}_{\rightarrow}$	$\mathbf{p}_{\mathrm{X}}=(\underline{\mathbf{C}}'_{\downarrow})^{-1}\mathbf{q}_{\mathrm{Z}}$, $\mathbf{q}_{\mathrm{X}}=\mathbf{e}_K$	$\mathbf{p}_{\mathrm{Z}}=\mathbf{e}_K$, $\mathbf{q}_{\mathrm{Z}}=v^*_{\mathrm{v}}$	$\mathbf{v}=\mathbf{v}_*$
(14)	$\underline{\mathbf{A}}_{\rightarrow}\underline{\mathbf{C}}_{\downarrow}$	$\mathbf{p}_{\mathrm{X}}=\mathbf{e}_K$, $\mathbf{q}_{\mathrm{X}}=\underline{\mathbf{C}}_{\rightarrow}^{-1}\mathbf{p}_{\mathrm{Z}}$	$\mathbf{p}_{\mathrm{Z}}=y^*_{\mathrm{y}}$, $\mathbf{q}_{\mathrm{Z}}=\mathbf{e}_K$	$\mathbf{y}=\mathbf{y}_*$
(15)	$\underline{\mathbf{A}}_{\rightarrow}\underline{\mathbf{C}}_{\rightarrow}$	$\mathbf{p}_{\mathrm{X}}=(\underline{\mathbf{X}}'_0-\underline{\mathbf{Z}}'_0)^{-1}\mathbf{v}_*$, $\mathbf{q}_{\mathrm{X}}=\mathbf{e}_K$	$\mathbf{p}_{\mathrm{Z}}=\mathbf{p}_{\mathrm{X}}$, $\mathbf{q}_{\mathrm{Z}}=\mathbf{e}_K$	$\mathbf{v}=\mathbf{v}_*$
(16)	$\underline{\mathbf{B}}_{\downarrow}\underline{\mathbf{D}}_{\downarrow}$	$\mathbf{p}_{\mathrm{X}}=\mathbf{e}_K$, $\mathbf{q}_{\mathrm{X}}=(\underline{\mathbf{X}}_0-\underline{\mathbf{Z}}_0)^{-1}\mathbf{y}_*$	$\mathbf{p}_{\mathrm{Z}}=\mathbf{e}_K$, $\mathbf{q}_{\mathrm{Z}}=\mathbf{q}_{\mathrm{X}}$	$\mathbf{y}=\mathbf{y}_*$
(17)	$\underline{\mathbf{B}}_{\downarrow}\underline{\mathbf{D}}_{\rightarrow}$	$\mathbf{p}_{\mathrm{X}}=\mathbf{e}_K$, $\mathbf{q}_{\mathrm{X}}=v^*_{\mathrm{v}}$	$\mathbf{p}_{\mathrm{Z}}=(\underline{\mathbf{D}}'_{\downarrow})^{-1}\mathbf{q}_{\mathrm{X}}$, $\mathbf{q}_{\mathrm{Z}}=\mathbf{e}_K$	$\mathbf{v}=\mathbf{v}_*$
(18)	$\underline{\mathbf{B}}_{\rightarrow}\underline{\mathbf{D}}_{\downarrow}$	$\mathbf{p}_{\mathrm{X}}=y^*_{\mathrm{y}}$, $\mathbf{q}_{\mathrm{X}}=\mathbf{e}_K$	$\mathbf{p}_{\mathrm{Z}}=\mathbf{e}_K$, $\mathbf{q}_{\mathrm{Z}}=\underline{\mathbf{D}}_{\rightarrow}^{-1}\mathbf{p}_{\mathrm{X}}$	$\mathbf{y}=\mathbf{y}_*$
(19)	$\underline{\mathbf{B}}_{\rightarrow}\underline{\mathbf{D}}_{\rightarrow}$	$\mathbf{p}_{\mathrm{X}}=(\underline{\mathbf{X}}'_0-\underline{\mathbf{Z}}'_0)^{-1}\mathbf{v}_*$, $\mathbf{q}_{\mathrm{X}}=\mathbf{e}_K$	$\mathbf{p}_{\mathrm{Z}}=\mathbf{p}_{\mathrm{X}}$, $\mathbf{q}_{\mathrm{Z}}=\mathbf{e}_K$	$\mathbf{v}=\mathbf{v}_*$

Again, the supplementary solutions for models $\underline{\mathbf{A}}_{\downarrow}\underline{\mathbf{C}}_{\downarrow}$ (12) and $\underline{\mathbf{B}}_{\downarrow}\underline{\mathbf{D}}_{\downarrow}$ (16) pairwise coincide as well as the supplementary solutions for models $\underline{\mathbf{A}}_{\rightarrow}\underline{\mathbf{C}}_{\rightarrow}$ (15) and $\underline{\mathbf{B}}_{\rightarrow}\underline{\mathbf{D}}_{\rightarrow}$ (19). Besides, there is only one vector in each pair $\mathbf{p}_{\mathrm{X}}$, $\mathbf{q}_{\mathrm{X}}$ that is distinct from summation vector $\mathbf{e}_K$. In the same way, a unique vector from each pair $\mathbf{p}_{\mathrm{Z}}$, $\mathbf{q}_{\mathrm{Z}}$ does not equal $\mathbf{e}_K$.

6. On the plausibility of regular and supplementary solutions obtained

An observing of the analytical results in Table 2 and 3 allows to recognize four different situations as follows:

$$\mathbf{X}=\hat{\mathbf{p}}_{\mathrm{X}}\mathbf{X}_0 \quad (\underline{\mathbf{X}}=\hat{\mathbf{p}}_{\mathrm{X}}\underline{\mathbf{X}}_0), \qquad \mathbf{Z}=\hat{\mathbf{p}}_{\mathrm{Z}}\mathbf{Z}_0 \quad (\underline{\mathbf{Z}}=\hat{\mathbf{p}}_{\mathrm{Z}}\underline{\mathbf{Z}}_0); \tag{22}$$

$$\mathbf{X}=\hat{\mathbf{p}}_{\mathrm{X}}\mathbf{X}_0 \quad (\underline{\mathbf{X}}=\hat{\mathbf{p}}_{\mathrm{X}}\underline{\mathbf{X}}_0), \qquad \mathbf{Z}=\mathbf{Z}_0\hat{\mathbf{q}}_{\mathrm{Z}} \quad (\underline{\mathbf{Z}}=\underline{\mathbf{Z}}_0\hat{\mathbf{q}}_{\mathrm{Z}}); \tag{23}$$

$$\mathbf{X}=\mathbf{X}_0\hat{\mathbf{q}}_{\mathrm{X}} \quad (\underline{\mathbf{X}}=\underline{\mathbf{X}}_0\hat{\mathbf{q}}_{\mathrm{X}}), \qquad \mathbf{Z}=\hat{\mathbf{p}}_{\mathrm{Z}}\mathbf{Z}_0 \quad (\underline{\mathbf{Z}}=\hat{\mathbf{p}}_{\mathrm{Z}}\underline{\mathbf{Z}}_0); \tag{24}$$

$$\mathbf{X}=\mathbf{X}_0\hat{\mathbf{q}}_{\mathrm{X}} \quad (\underline{\mathbf{X}}=\underline{\mathbf{X}}_0\hat{\mathbf{q}}_{\mathrm{X}}), \qquad \mathbf{Z}=\mathbf{Z}_0\hat{\mathbf{q}}_{\mathrm{Z}} \quad (\underline{\mathbf{Z}}=\underline{\mathbf{Z}}_0\hat{\mathbf{q}}_{\mathrm{Z}}). \tag{25}$$

Situation (22) describes an impact of exogenous changes in final demand or value added on the economy in terms of price changing exclusively, whereas situation (25) characterizes this one in terms of volume changing only. Mixed situations (23) and (24) combine price and volume changes and in the presence of the detected linkages between vectors $\mathbf{p}_{\mathbf{X}}, \mathbf{q}_{\mathbf{X}}$ and $\mathbf{p}_{\mathbf{Z}}, \mathbf{q}_{\mathbf{Z}}$ (see models $\mathbf{A}_{\downarrow}\mathbf{C}_{\rightarrow}$, $\mathbf{A}_{\rightarrow}\mathbf{C}_{\downarrow}$, $\mathbf{B}_{\downarrow}\mathbf{D}_{\rightarrow}$, $\mathbf{B}_{\rightarrow}\mathbf{D}_{\downarrow}$ in Table 2 and models $\underline{\mathbf{A}}_{\downarrow}\underline{\mathbf{C}}_{\rightarrow}$, $\underline{\mathbf{A}}_{\rightarrow}\underline{\mathbf{C}}_{\downarrow}$, $\underline{\mathbf{B}}_{\downarrow}\underline{\mathbf{D}}_{\rightarrow}$, $\underline{\mathbf{B}}_{\rightarrow}\underline{\mathbf{D}}_{\downarrow}$ in Table 3) seem to be implausible artefacts that are out of economic sense. By this reason the models (13), (14), (17), (18) are not examined further, and the corresponding rows in Table 2 and 3 are darkened.

It is interesting to note here that models $\mathbf{A}_{\downarrow}\mathbf{C}_{\rightarrow}$ and $\underline{\mathbf{A}}_{\downarrow}\underline{\mathbf{C}}_{\rightarrow}$ generate an instrumental framework for an industry technology assumption and a fixed product sales structure assumption, which are widely used in the transformation of supply and use tables to symmetric input-output tables (see Eurostat, 2008).

7. The Leontief and Ghosh quantity and price models

It is easy to see that regular solutions for models (12), (15), (16), (19) are trivial, but their corresponding supplementary solutions are quite not.

Model $\underline{\mathbf{A}}_{\downarrow}\underline{\mathbf{C}}_{\downarrow}$ (12) is well-known as a Leontief demand-driven model. It serves to assess an impact of exogenous (absolute or relative) changes in final demand on the economy at fixed prices. Indeed, the main model's statements are $\underline{\mathbf{X}} = \underline{\mathbf{X}}_0\hat{\mathbf{q}}$ and $\underline{\mathbf{Z}} = \underline{\mathbf{Z}}_0\hat{\mathbf{q}}$ where

$$\mathbf{q} = \left(\underline{\mathbf{X}}_0 - \underline{\mathbf{Z}}_0\right)^{-1}\mathbf{y}_* = \left(\underline{\mathbf{C}}_{\rightarrow} - \underline{\mathbf{A}}_{\rightarrow}\right)^{-1}\left\langle\underline{\mathbf{X}}_0\mathbf{e}_K\right\rangle^{-1}\mathbf{y}_* = \left(\underline{\mathbf{C}}_{\rightarrow} - \underline{\mathbf{A}}_{\rightarrow}\right)^{-1}\boldsymbol{y}_x\,. \tag{26}$$

Model $\underline{\mathbf{A}}_{\rightarrow}\underline{\mathbf{C}}_{\rightarrow}$ (15) is known as a Ghosh supply-driven model. It helps to evaluate an impact of exogenous (absolute or relative) changes in value added on the economy at fixed production scales. The main model's statements are $\underline{\mathbf{X}} = \hat{\mathbf{p}}\underline{\mathbf{X}}_0$ and $\underline{\mathbf{Z}} = \hat{\mathbf{p}}\underline{\mathbf{Z}}_0$ where

$$\mathbf{p} = \left(\underline{\mathbf{X}}'_0 - \underline{\mathbf{Z}}'_0\right)^{-1}\mathbf{v}_* = \left(\underline{\mathbf{C}}'_{\downarrow} - \underline{\mathbf{A}}'_{\downarrow}\right)^{-1}\left\langle\underline{\mathbf{X}}_0\mathbf{e}_K\right\rangle^{-1}\mathbf{v}_* = \left(\underline{\mathbf{C}}'_{\downarrow} - \underline{\mathbf{A}}'_{\downarrow}\right)^{-1}\boldsymbol{v}_x\,. \tag{27}$$

In a symmetric input-output table, initial production matrix $\underline{\mathbf{X}}_0$ is a diagonal matrix and therefore

$$\underline{\mathbf{X}}_0 = \underline{\mathbf{X}}'_0 = \left\langle\mathbf{e}'_K\underline{\mathbf{X}}_0\right\rangle = \left\langle\underline{\mathbf{X}}_0\mathbf{e}_K\right\rangle. \tag{28}$$

Using (26) and then (28), we obtain famous Leontief formula

$$\mathbf{x}_{\downarrow} = \mathbf{x}_{\rightarrow} = \underline{\mathbf{X}}'\mathbf{e}_K = \hat{\mathbf{q}}\underline{\mathbf{X}}'_0\mathbf{e}_K = \left\langle\underline{\mathbf{X}}'_0\mathbf{e}_K\right\rangle\left(\underline{\mathbf{X}}_0 - \underline{\mathbf{Z}}_0\right)^{-1}\mathbf{y}_* = \left[\left(\underline{\mathbf{X}}_0 - \underline{\mathbf{Z}}_0\right)\left\langle\mathbf{e}'_K\underline{\mathbf{X}}_0\right\rangle^{-1}\right]^{-1}\mathbf{y}_* = \left(\mathbf{E}_K - \underline{\mathbf{A}}_{\downarrow}\right)^{-1}\mathbf{y}_*\,,$$

and from (27) with (28) we find its Ghosh supply-driven analogue

$$\mathbf{x}_{\rightarrow} = \mathbf{x}_{\downarrow} = \underline{\mathbf{X}}\mathbf{e}_K = \hat{\mathbf{p}}\underline{\mathbf{X}}_0\mathbf{e}_K = \left\langle\underline{\mathbf{X}}_0\mathbf{e}_K\right\rangle\left(\underline{\mathbf{X}}'_0 - \underline{\mathbf{Z}}'_0\right)^{-1}\mathbf{v}_* = \left[\left(\underline{\mathbf{X}}'_0 - \underline{\mathbf{Z}}'_0\right)\left\langle\underline{\mathbf{X}}_0\mathbf{e}_K\right\rangle^{-1}\right]^{-1}\mathbf{v}_* = \left(\mathbf{E}_K - \underline{\mathbf{A}}'_{\rightarrow}\right)^{-1}\mathbf{v}_*\,.$$

Putting (28) into Ghosh model (27) gives well-known formula

$$\mathbf{p} = \left(\underline{\mathbf{X}}'_0 - \underline{\mathbf{Z}}'_0\right)^{-1}\mathbf{v}_* = \left(\left\langle \mathbf{e}'_K \underline{\mathbf{X}}_0 \right\rangle - \underline{\mathbf{Z}}'_0\right)^{-1}\mathbf{v}_* = \left(\mathbf{E}_K - \underline{\mathbf{A}}'_{\downarrow}\right)^{-1}\left\langle \mathbf{e}'_K \underline{\mathbf{X}}_0 \right\rangle^{-1}\mathbf{v}_* = \left(\mathbf{E}_K - \underline{\mathbf{A}}'_{\downarrow}\right)^{-1}\boldsymbol{v}_x$$

for Leontief price model (see Miller and Blair, 2009, p. 44). Thus, in the case of a symmetric input-output table Ghosh supply-driven model coincides with Leontief price model (see Dietzenbacher, 1997). It can be shown by analogy that Leontief demand-driven model serves as Ghosh quantity model. Indeed, putting (28) into Leontief model (26) gives

$$\mathbf{q} = \left(\underline{\mathbf{X}}_0 - \underline{\mathbf{Z}}_0\right)^{-1}\mathbf{y}_* = \left(\left\langle \underline{\mathbf{X}}_0 \mathbf{e}_K \right\rangle - \underline{\mathbf{Z}}_0\right)^{-1}\mathbf{y}_* = \left(\mathbf{E}_K - \underline{\mathbf{A}}_{\rightarrow}\right)^{-1}\left\langle \underline{\mathbf{X}}_0 \mathbf{e}_K \right\rangle^{-1}\mathbf{y}_* = \left(\mathbf{E}_K - \underline{\mathbf{A}}_{\rightarrow}\right)^{-1}\boldsymbol{y}_x .$$

It is appropriate mention here that all formulae obtained above demonstrate a remarkable set of duality properties.

8. Concluding remarks

As noted earlier, both regular and supplementary solutions for models (12) and (16) pairwise coincide as well as respective solutions for models (15) and (19). Thus, models $\underline{\mathbf{A}}_{\downarrow}\underline{\mathbf{C}}_{\downarrow}$, $\underline{\mathbf{B}}_{\downarrow}\underline{\mathbf{D}}_{\downarrow}$ and $\underline{\mathbf{A}}_{\rightarrow}\underline{\mathbf{C}}_{\rightarrow}$, $\underline{\mathbf{B}}_{\rightarrow}\underline{\mathbf{D}}_{\rightarrow}$ seem to be pairwise equivalent from a computing viewpoint.

Recall that Leontief model (12) is based on the matrix-valued cost function (4) and the product (“vertical”) structure of production matrix (8), whereas a framework of Ghosh model (15) envelops the matrix-valued cost functions in another form (5) and the industry (“horizontal”) structure of production matrix (9). Matrices $\underline{\mathbf{A}}_{\downarrow}$ and $\underline{\mathbf{A}}_{\rightarrow}$ are known in special literature as (Leontief) technical coefficients matrix and (Ghosh) allocation coefficients matrix (see Miller and Blair, 2009) respectively.

In turn, model (16) is provided by the matrix-valued production function (6) and the product (“vertical”) structure of intermediate consumption matrix (10), whereas model (19) is based on the matrix-valued production functions in another form (7) and the industry (“horizontal”) structure of intermediate consumption matrix (11). Hence, the elements of matrices $\underline{\mathbf{B}}_{\downarrow}$ and $\underline{\mathbf{B}}_{\rightarrow}$ are “quasi-reciprocal” with respect to technical coefficients matrix $\underline{\mathbf{A}}_{\downarrow}$ and allocation coefficients matrix $\underline{\mathbf{A}}_{\rightarrow}$ respectively.

Nevertheless, practical applying the models $\underline{\mathbf{A}}_{\downarrow}\underline{\mathbf{C}}_{\downarrow}$ and $\underline{\mathbf{B}}_{\downarrow}\underline{\mathbf{D}}_{\downarrow}$ as well as $\underline{\mathbf{A}}_{\rightarrow}\underline{\mathbf{C}}_{\rightarrow}$ and $\underline{\mathbf{B}}_{\rightarrow}\underline{\mathbf{D}}_{\rightarrow}$ leads to identical results (see Table 2 and 3). This means that certain choice of the coefficients matrix forms does not have an influence on results of modelling. Here one can really solve a dilemma of taking “vertical” or “horizontal” structure for constructing a dependency between the production matrix $\mathbf{X}$ and the intermediate consumption matrix $\mathbf{Z}$ resembling (4) – (7). Thus, technical and allocation coefficients should be regarded as helpful ways of economic interpretation rather than as operational tools for calculation.

References

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